

Resilient pancyclicity of random and pseudo-random graphs

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Abstract

A graph G on n vertices is *pancyclic* if it contains cycles of length t for all $3 \leq t \leq n$. In this paper we prove that for any fixed $\epsilon > 0$, the random graph $G(n, p)$ with $p(n) \gg n^{-1/2}$ asymptotically almost surely has the following resilience property. If H is a subgraph of G with maximum degree at most $(1/2 - \epsilon)np$ then $G - H$ is pancyclic. In fact, we prove a more general result which says that if $p \gg n^{-1+1/(l-1)}$ for some integer $l \geq 3$ then for any $\epsilon > 0$, asymptotically almost surely every subgraph of $G(n, p)$ with minimum degree greater than $(1/2 + \epsilon)np$ contains cycles of length t for all $l \leq t \leq n$. These results are tight in two ways. First, the condition on p essentially cannot be relaxed. Second, it is impossible to improve the constant $1/2$ in the assumption for the minimum degree. We also prove corresponding results for pseudo-random graphs.

1 Introduction

A typical result in graph theory can be stated as “Under certain conditions, a graph G possesses a property \mathcal{P} ”. Once this type of a result is established, it is natural to ask “How strongly does G possess \mathcal{P} ?”. In fact, several important results in extremal graph theory can be viewed as an answer to this question for various graph properties (we will provide some concrete examples after introducing necessary definitions). In this paper we will study this question in the context of random and pseudo-random graphs. The random graph model we consider is the binomial random graph $G(n, p)$. The random graph $G(n, p)$ denotes the probability space whose points are graphs with vertex set $[n] = \{1, \dots, n\}$ where each pair of vertices forms an edge randomly and independently with probability p . We say that $G(n, p)$ possesses a graph property \mathcal{P} *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G(n, p)$ possesses \mathcal{P} tends to 1 as n goes to infinity. The pseudo-random graphs we will study are (n, d, λ) -graphs with $\lambda = o(d)$, where an (n, d, λ) -graph is a d -regular graph on n vertices whose second largest (in absolute value) eigenvalue of the adjacency matrix is bounded by λ . The abundance of structure and results arising from this simple looking definition is quite surprising (see, e.g., [13] for more details). A graph property is called *monotone increasing* (*decreasing*) if it is preserved under edge addition (deletion).

The main concept studied in this paper and briefly outlined above is that of *resilience*. Formally, following [16], we define:

Definition 1. Let \mathcal{P} be a monotone increasing (decreasing) graph property.

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- (i) (Global resilience) The global resilience of G with respect to \mathcal{P} is the minimum number r such that by deleting (adding) r edges from G one can obtain a graph not having \mathcal{P} .
- (ii) (Local resilience) The local resilience of a graph G with respect to \mathcal{P} is the minimum number r such that by deleting (adding) at most r edges at each vertex of G one can obtain a graph not having \mathcal{P} .

Using this terminology, one can state the celebrated theorem of Turán [17] as “The complete graph on n vertices K_n has global resilience $\frac{n^2}{2r} - \frac{n}{2}$ with respect to being K_{r+1} -free”. Another classical theorem, that of Dirac (see, e.g., [7]) can be rephrased as “ K_n has local resilience $\lfloor n/2 \rfloor$ with respect to Hamiltonicity”. As these examples suggest, the notion of resilience lies in the center of extremal graph theory. In [16], Sudakov and Vu have initiated the systematic study of global and local resilience of random and pseudo-random graphs. They obtained resilience results with respect to various properties such as perfect matching, hamiltonicity, chromatic number and having a nontrivial automorphism (this result appeared in their earlier paper with Kim [12]). For example, they showed that if $p > \log^4 n/n$ then a.a.s. any subgraph of $G(n, p)$ with minimum degree $(1/2 + o(1))np$ is hamiltonian. An interesting thing to notice is that this result can be viewed as a generalization of Dirac’s Theorem mentioned above as a complete graph is also a random graph $G(n, p)$ with $p = 1$. As we will see, this connection is very natural and most of the resilience results can be viewed as a generalization of classic graph theory results to random and pseudo-random graphs.

There are several other papers that obtained resilience type results. Krivelevich and Frieze [8] gave a lower bound (not tight) on resilience of $G(n, p)$ with respect to being Hamiltonian in the range of p not covered by the above mentioned result of Sudakov and Vu. Dellamonica, Kohayakawa, Marciniszyn and Steger [6] studied the global resilience of random graphs with respect to containing a cycle of length at least $(1 - \alpha)n$ for a fixed α as a generalization of a theorem of Woodall [18]. Recently, Ben-Shimon, Krivelevich and Sudakov [2] investigated the resilience of random regular graphs with respect to being hamiltonian.

A graph on n vertices is called *pancyclic* if it contains cycles of length t for all $3 \leq t \leq n$. In this paper, we study the resilience of random and pseudo-random graphs with respect to this property. Similarly to the above mentioned results, our result can also be viewed as a generalization of a classical result in graph theory – that by Bondy [3]. It says that if G is a graph on n vertices with minimum degree greater than $n/2$, then G is pancyclic. The corresponding theorems we prove are,

Theorem 1.1. *If $p \gg n^{-1/2}$ then $G(n, p)$ asymptotically almost surely has local resilience $(1/2 + o(1))np$ with respect to being pancyclic.*

Theorem 1.2. *Let $G = (V, E)$ be a (n, d, λ) -graph satisfying $d^2/n \gg \lambda$. Then G has local resilience $(1/2 + o(1))d$ with respect to being pancyclic.*

Our results are asymptotically tight in two ways. First, one cannot improve the constant $1/2$ since both random and pseudo-random graphs can be made bipartite by randomly partitioning the graph into two equal size parts. In this way we typically have a subgraph with minimum degree about one half of the original degree which does not contain any odd cycles. Second, the restrictions on the parameters are also essentially tight. To see this for random graphs, note that if $p \ll n^{-1/2}$ then typically each vertex has degree $(1 + o(1))np$ and the number of triangles containing each vertex is at most $O(n^2 p^3) \ll np$. Therefore deleting edges of all triangles leaves all degrees essentially unchanged.

For pseudo-random graphs this can be derived from a variant of the construction of Alon [1] (see, e.g., [13]) which gives a triangle-free (n, d, λ) -graph with $d^3/\lambda^2 = \Theta(n)$.

We can also prove more general results for sparser graphs. Let the *girth* of a graph be the length of its shortest cycle and the *circumference* be the length of its longest cycle. Brandt, Faudree and Goddard [5] called a graph *weakly pancyclic* if it contains cycles of length t where t ranges from its girth up to its circumference. The following theorems, generalizing Theorems 1.1 and 1.2, are motivated by this concept of weak pancyclicity.

Theorem 1.3. *For any fixed integer $l \geq 3$, if $p \gg n^{-1+1/(l-1)}$ then $G(n, p)$ asymptotically almost surely has local resilience $(1/2 + o(1))np$ with respect to containing cycles of length t for all $l \leq t \leq n$.*

Theorem 1.4. *Let k be either 3 or an even integer satisfying $k \geq 4$ and let $G = (V, E)$ be a (n, d, λ) -graph satisfying $d^{k-1}/n \gg \lambda^{k-2}$. Then G has local resilience $(1/2 + o(1))d$ with respect to containing cycles of length t for all $k \leq t \leq n$.*

The above results are not exactly weak pancyclicity results since if we allow the adversary to delete half of the edges at each vertex he might decide not to remove a 3-cycle and then remove every other cycle of length 4 up to $l - 1$. But still it is best to view these results in the context of weak pancyclicity. Similarly as before, the result for random graphs is asymptotically tight. Indeed, note that if $p \ll n^{-1+1/(l-1)}$ then typically each vertex of the random graph has degree $(1 + o(1))np$ and the number of cycles of length l containing each vertex is at most $O(n^{l-1}p^l) \ll np$. Therefore we can delete few edges from each vertex to remove every l -cycle. We suspect that our result for pseudo-random graphs is asymptotically tight as well. Note that the assumption $d^{k-1}/n \gg \lambda^{k-2}$ in particular implies $\lambda = o(d)$ since $d^{k-2} \geq d^{k-1}/n \gg \lambda^{k-2}$, so even when we do not explicitly mention $\lambda = o(d)$, we are always in this situation. Although odd integers $k > 3$ are omitted from the result of pseudo-random graphs, nevertheless in this case $d^{k-1}/n \gg \lambda^{k-2}$ implies $d^k/n \gg \lambda^{k-1}$ and so by using the result for $k + 1$ (which is now even) we can find cycles of length t for all $k + 1 \leq t \leq n$. We believe that the result of Theorem 1.4 is valid also for odd $k \geq 5$, but at present we do not have enough tools to verify it. We will address this point in more details in concluding remarks.

The rest of this paper is organized as follows. In Section 2 we collect some known results which we need later to prove our main theorems. In Section 3 we establish properties of random graphs and use them in Section 4 to prove Theorem 1.3. In Sections 5, 6 we follow the same pattern to prove the pseudo-random graph analog, Theorem 1.4. The last section contains some concluding remarks and open problems.

Notation. $G = (V, E)$ denotes a graph with vertex set V and edge set E . We use $v \sim w$ to indicate that v, w are adjacent. $\Delta(G), \delta(G)$ denote the maximum degree and the minimum degree of G , respectively. For a set $X \subset V$, let $N(X)$ be the collection of all vertices v which are adjacent to at least one vertex in X . If $X = \{u\}$ is a singleton set we denote its neighborhood by $N(u)$. Let $N^{(0)}(v) := \{v\}$ and $N^{(k)}(v)$ be the vertices at distance exactly k from v . This can also be recursively defined as $N^{(k)}(v) = N(N^{(k-1)}(v)) \setminus (N^{(k-1)}(v) \cup N^{(k-2)}(v))$. Note that $N^{(1)}(v) = N(v)$. For a set X , we denote by $E(X)$ the set of edges in the induced subgraph $G[X]$ and by $e(X) = |E(X)|$ its size. Similarly, for two sets X and Y , we denote by $E(X, Y)$ the set of ordered pairs $(x, y) \in E$ such that $x \in X$ and $y \in Y$, also $e(X, Y) = |E(X, Y)|$. Note that $e(X, X) = 2e(X)$. If we have several graphs, then the graph we are currently working with will be stated as a subscript. For example $N_G^{(k)}(v)$ is the k -th neighborhood of v in graph G . A cycle of length l is denoted by C_l .

We also utilize the following standard asymptotic notation. For two functions $f(n)$ and $g(n)$, write $f(n) = \Omega(g(n))$ if there exists a constant C such that $\liminf_{n \rightarrow \infty} f(n)/g(n) \geq C$. If there is a subscript such as in Ω_ϵ this means that the constant C may depend on ϵ . We write $f(n) = o(g(n))$ or $f(n) \ll g(n)$ if $\limsup_{n \rightarrow \infty} f(n)/g(n) = 0$. Also, $f(n) = O(g(n))$ if there exists a positive constant $C > 0$ such that $\limsup_{n \rightarrow \infty} f(n)/g(n) \leq C$. Throughout the paper \log denotes the natural logarithm. To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize absolute constants involved. We also assume that the order n of all graphs tends to infinity and therefore is sufficiently large whenever necessary.

2 Preliminaries

In this section we collect various results to be used later in the proofs of the theorems.

2.1 Resilience

The local resilience of random graphs with respect to being hamiltonian [16] and containing fixed cycles ([9], [10], [15]) have been studied before and our arguments for the proof of the main theorems will use these results. The following results about the local resilience of random and pseudo-random graphs with respect to hamiltonicity were proved in [16].

Theorem 2.1. *For every fixed $\epsilon > 0$, if $p \geq \log^4 n/n$ then the random graph $G(n, p)$ with probability $1 - o(n^{-1})$ has local resilience at least $(1/2 - \epsilon)np$ with respect to being hamiltonian.*

Remark. The above formulation is stronger than the original statement since it explicitly states the success probability to be $1 - o(n^{-1})$. But this conclusion follows from the original argument if one carefully performs the error probability calculations. We will need this stronger estimate on success probability for our application.

During the proof we will work with graphs that are similar to (n, d, λ) -graphs but are not necessarily regular. The particular graphs we will encounter are graphs $G = (V, E)$ on n vertices that have minimum degree at least $(1 - \epsilon)d$ and satisfy the constraint

$$|e(X, Y) - \frac{d}{n}|X||Y|| \leq \lambda\sqrt{|X||Y|} \text{ for all } X, Y \subset V$$

on the number of edges between sets. We will call such graphs $(n, \epsilon, d, \lambda)$ -graphs. Observe that $(n, 0, d, \lambda)$ -graphs are a more general/flexible concept than that of (n, d, λ) -graphs, as it does not put specific assumptions on graph eigenvalues.

Theorem 2.2. *Fix ϵ, ϵ' such that $0 \leq 5\epsilon' < \epsilon, k \geq 3$ and let G be an $(n, \epsilon', d, \lambda)$ -graph satisfying $d^{k-1}/n = \omega(n)\lambda^{k-2}$ for an arbitrary function $\omega(n)$ increasing to infinity. Then for large enough n , G has local resilience at least $(1/2 - \epsilon)d$ with respect to being hamiltonian.*

Remark. The theorem above does not appear in the original paper [16] and unfortunately we cannot directly apply the result from Sudakov and Vu. But in fact, they proved a general theorem which can be modified to work under the assumption above. The necessarily modification will be given in the Appendix.

Next we state the results of Haxell, Kohayakawa and Łuczak ([9], [10]) about the local resilience of random graphs with respect to containing a fixed cycle C_l , and of Sudakov, Szabò and Vu [15] about the local resilience of pseudo-random graphs with respect to containing a triangle.

Theorem 2.3. *For any fixed integer $l \geq 3$ and $\epsilon > 0$, there exists a constant $C = C(l, \epsilon)$ such that, if $p \geq Cn^{-1+1/(l-1)}$ then $G(n, p)$ a.a.s. has local resilience at least $(1/2 - \epsilon)np$ with respect to containing C_l .*

Theorem 2.4. *Let G be a (n, d, λ) -graph satisfying $d^2/n \geq \omega(n)\lambda$ for an arbitrary function $\omega(n)$ tending to infinity. Then G has local resilience $(1/2 + o(1))d$ with respect to containing a triangle.*

Remark. Both theorems are originally stated in a global resilience form but for convenience we stated it as above in a slightly weaker local resilience form. Also the conclusion of Theorem 2.3 (as stated) for even cycles is weaker than in the original paper.

2.2 Extremal Graph Theory

The following simple but useful lemma allows one to find a large minimum degree subgraph in a graph with large average degree. (See, e.g., [7], Proposition 1.2.2)

Lemma 2.5. *Let $G = (V, E)$ be a graph on n vertices with at least $dn/2$ edges. Then G contains a subgraph $G' \subset G$ with minimum degree at least $d/2$.*

Next theorem is a classical result by Bondy and Simonovits [4] about even cycles in graphs.

Theorem 2.6. *Let k be a positive integer and $G = (V, E)$ be a graph on n vertices satisfying $|E| > 90kn^{1+1/k}$. Then G contains a cycle of length $2k$.*

We will also need the celebrated Pòsa rotation-extension lemma (see [14], Ch. 10, Problem 20). This lemma will help us in finding long paths in a graph with expansion properties.

Lemma 2.7. *Let $G = (V, E)$ be a graph such that $|N(X) \setminus X| \geq 2|X| - 1$ for all $X \subset V$ with $|X| \leq t$. Then for any vertex $v \in V$ there exists a path of length $3t - 2$ in G that has v as an end point.*

2.3 Concentration

The following two well-known concentration results (see, for example [11], Theorems 2.3 and 2.10) will be used several times during the proof. We denote by $Bi(n, p)$ a binomial random variable with parameters n and p .

Theorem 2.8. (Chernoff inequality) *If $X \sim Bi(n, p)$ and $\epsilon > 0$, then*

$$P(|X - \mathbb{E}[X]| \geq \epsilon \mathbb{E}[X]) \leq e^{-\Omega_\epsilon(\mathbb{E}[X])}.$$

Let m, n and N be positive integers with $m, n < N$, let $X = [N]$, $X' = [n]$, and let A be a m -element subset of X chosen uniformly at random. Then the distribution of the random variable $|A \cap X'|$ is called the *hypergeometric distribution* with parameters N, n and m .

Theorem 2.9. *Let X have the hypergeometric distribution with parameters N, n and m . Then,*

$$P(|X - \mathbb{E}[X]| \geq \epsilon \mathbb{E}[X]) \leq e^{-\Omega_\epsilon(\mathbb{E}[X])}.$$

3 Properties of Random Graphs

In this section we establish properties of random graphs to be used later to prove Theorem 1.3.

First we show formally a rather expected monotonicity property – (relative) local resilience with respect to cycles can only grow with the edge probability $p(n)$.

Proposition 3.1. *Let l be fixed and let $p' = p'(n)$ satisfy: $0 < p' \leq p \leq 1$ and $np' \gg \log n$. If $G(n, p')$ a.a.s. has local resilience at least $(1/2 - \epsilon/2)np'$ with respect to containing cycles of length t for all $l \leq t \leq n$ then $G(n, p)$ a.a.s. has local resilience at least $(1/2 - \epsilon)np$ with respect to the same property.*

Proof. Let \mathcal{P} be the property of having local resilience at least $(1/2 - \epsilon/2)np'$ with respect to containing every cycles of length t for all $l \leq t \leq n$. Define $q = p'/p$ and consider the following two round process of exposing the edges of $G(n, p')$. In the first round, every edge appears with probability p (call this graph G_1). Then at the second round, every edge that appeared in the first round will remain with probability q and will be deleted with probability $1 - q$ (call this graph G_2). Then G_1 has the same distribution as $G(n, p)$ and G_2 has the same distribution as $G(n, p')$. By our assumption we know that G_2 a.a.s. has property \mathcal{P} . Now define X to be the event that G_1 satisfies: $P(G_2 \notin \mathcal{P} | G_1) \geq 1/2$. Then $(1/2)P(X) \leq P(G_2 \notin \mathcal{P}) = o(1)$ and therefore $P(X) = o(1)$. Thus a.a.s. in $G(n, p)$, $P(G_2 \notin \mathcal{P} | G_1) < 1/2$ or in other words $P(G_2 \in \mathcal{P} | G_1) \geq 1/2$. Let \mathcal{A} be the collection of graphs $G_1 \in G(n, p)$ having this property.

Now given any subgraph H of G_1 with maximum degree at most $(1/2 - \epsilon)np$, select every edge with probability q to get a graph H' . Then by Chernoff inequality, each vertex of H' has maximum degree at most $(1/2 - \epsilon/2)np'$ with probability at least $1 - e^{-\Omega_\epsilon(np')} = 1 - o(n^{-1})$. Therefore H' has maximum degree at most $(1/2 - \epsilon/2)np'$ with probability at least $1 - o(1)$.

Finally to put things together, condition on the event that $G_1 = G(n, p) \in \mathcal{A}$. By the first part of the proof a.a.s. $G \in \mathcal{A}$ so if we can prove the claim under this assumption then we are done. Given a subgraph $H \subset G$ with maximum degree at most $(1/2 - \epsilon)np$, sample every edge of G with probability q to obtain subgraphs $H' \subset G' \subset G$. Since $G \in \mathcal{A}$, we know that $P(G' \in \mathcal{P} | G) \geq 1/2$ and by the second part of the proof we know that $P(\Delta(H') \leq (1/2 - \epsilon/2)np') \geq 1 - o(1)$. Thus, these two events have a non-empty intersection and therefore it is possible to find subgraphs $H' \subset G' \subset G$ such that $G' \in \mathcal{P}$ and $\Delta(H') \leq (1/2 - \epsilon/2)np'$. Then $G' - H'$ (and hence $G - H$) must contain cycles of length t for all $l \leq t \leq n$. \square

Remark. Note that there is nothing special about the property of “containing cycles” and in fact if for some $\log n/n \ll p' \leq p$ and $\alpha > 0$, we have a monotone increasing graph property \mathcal{Q} such that $G(n, p')$ a.a.s. has local resilience at least $(\alpha + o(1))np'$ with respect to having property \mathcal{Q} then $G(n, p)$ a.a.s. has local resilience at least $(\alpha + o(1))np$ with respect to having property \mathcal{Q} .

From now on we may assume that $p = Cn^{-1+1/(l-1)}$ instead of $p \geq Cn^{-1+1/(l-1)}$ since if we can prove the theorem under this condition then we can extend it to the whole range using the previous proposition. Moreover we will assume that the constant C is large enough without further mentioning.

In the next two lemmas we establish some expansion properties of random graphs.

Lemma 3.2. *Fix a positive integer l and $0 < \epsilon < 1$ and let $G = (V, E)$ be a random graph $G(n, p)$ with $p = Cn^{-1+1/(l-1)}$. Then a.a.s. every subset $X \subset V$ of size $|X| \leq (2C)^{l-1}n^{-1}p^{-2}$ satisfies $(1 - \epsilon)|X|np \leq |N(X)| \leq (1 + \epsilon)|X|np$.*

Proof. Fix a set $X \subset V$ of size $|X| \leq (2C)^{l-1}n^{-1}p^{-2}$. For each $v \in V$ let Y_v be indicator random variable of the event that $v \in N(X)$. Since $|X|p = o(1)$, we have $P(Y_v = 1) = 1 - (1 - p)^{|X|} = (1 + o(1))|X|p$. Consider the random variable $Y = \sum_{v \in V \setminus X} Y_v = |N(X) \setminus X|$ and note that

$$\mathbb{E}[Y] = \sum_{v \in V \setminus X} P(Y_v = 1) = (n - |X|)(1 + o(1))|X|p = (1 + o(1))|X|np.$$

Moreover if $v \in V \setminus X$ then Y_v are mutually independent so we can apply the Chernoff inequality to get

$$P(|Y - \mathbb{E}[Y]| \geq (\epsilon/3)\mathbb{E}[Y]) \leq e^{-\Omega_\epsilon(\mathbb{E}[Y])}.$$

Combine this with the estimate on $\mathbb{E}[Y]$ and we have,

$$P((1 - 2\epsilon/3)|X|np \leq Y \leq (1 + 2\epsilon/3)|X|np) \geq 1 - e^{-\Omega_\epsilon(\mathbb{E}[Y])} = 1 - e^{-\Omega_\epsilon(|X|np)}$$

for large enough n . Finally, note that $|N(X) - Y| \leq |X| = o(|X|np)$ and thus for large enough n , we have $(1 - \epsilon)|X|np \leq |N(X)| \leq (1 + \epsilon)|X|np$ with probability at least $1 - e^{-\Omega_\epsilon(|X|np)}$.

Taking the union bound over all choices of X , we get

$$\begin{aligned} \sum_{1 \leq |X| \leq (2C)^{l-1}n^{-1}p^{-2}} e^{-\Omega_\epsilon(|X|np)} &= \sum_{1 \leq k \leq (2C)^{l-1}n^{-1}p^{-2}} \binom{n}{k} e^{-\Omega_\epsilon(knp)} \leq \sum_{1 \leq k \leq (2C)^{l-1}n^{-1}p^{-2}} \left(\frac{en}{k} e^{-\Omega_\epsilon(np)} \right)^k \\ &\leq \sum_{1 \leq k \leq (2C)^{l-1}n^{-1}p^{-2}} ne^{-\Omega_\epsilon(np)} \leq n^2 e^{-\Omega_\epsilon(np)} = o(1). \end{aligned}$$

This implies the assertion of the lemma. \square

Lemma 3.3. Fix a positive integer l and $0 < \epsilon < 1$ and let $G = G(n, p)$ be a random graph with $p = Cn^{-1+1/(l-1)}$. Then a.a.s. G has the following property. If H is a subgraph of G with maximum degree at most $(1/2 - \epsilon)np$ and $G' = G - H$, then every set X with $|X| \geq 2^{-l}(np)^{l-2}$ satisfies $|N_{G'}(X)| \geq (1/2 + \epsilon/2)n$.

Proof. It is enough to show that a.a.s. for any H as above the claim holds for every set X with size exactly $|X| = 2^{-l}(np)^{l-2}$. Fix a set X of size $2^{-l}(np)^{l-2}$ and let $Y \subset V$ be a set of size $|Y| \geq (1/2 - \epsilon/2)n$ disjoint from X . Then we have $\mathbb{E}[e_G(X, Y)] = |X||Y|p > 2^{-l-2}(np)^{l-1} = 2^{-l-2}C^{l-1}n$ and by the Chernoff inequality,

$$P\left(|e_G(X, Y) - |X||Y|p| \geq (\epsilon/4)|X||Y|p\right) < e^{-\Omega_\epsilon(|X||Y|p)} \leq e^{-\Omega_\epsilon(2^{-l-2}C^{l-1}n)}. \quad (1)$$

Thus with probability at least the right hand side of (1) we have $e_G(X, Y) \geq (1 - \epsilon/4)|X||Y|p \geq (\frac{1}{2} - 3\epsilon/4)|X|np$. Since there are at most 2^{2n} possible choices of the pairs X, Y and the right hand side of (1) is $\ll 2^{-2n}$ for large enough C , we a.a.s. have $e_G(X, Y) > (\frac{1}{2} - \epsilon)|X|np$ for every pair X, Y as above.

On the other hand, we know that $e_H(X, Y) \leq (1/2 - \epsilon)np|X|$. Therefore a.a.s. $e_{G'}(X, Y) \geq e_G(X, Y) - e_H(X, Y) > 0$. This implies that $N_{G'}(X) \cap Y \neq \emptyset$ for all Y with $|Y| \geq (1/2 - \epsilon/2)n$. Thus $|N_{G'}(X)| \geq n - (1/2 - \epsilon/2)n = (1/2 + \epsilon/2)n$. \square

We also need the following lemma that proves expansion property for subgraphs of $G(n, p)$ with large minimum degree.

Lemma 3.4. *If $p = Cn^{-1+1/(l-1)}$ and $\epsilon' > 0$ then a.a.s. every subgraph $G' \subset G(n, p)$ with minimum degree at least $\epsilon'np$ satisfies the following expansion property. For all $X \subset V$ with $|X| \leq \frac{1}{80}\epsilon'n$, $|N_{G'}(X) \setminus X| \geq 2|X|$.*

Proof. Assume to the contrary that there exists a set $X \subset V$ such that $|X| \leq \frac{1}{80}\epsilon'n$ and $|N_{G'}(X) \setminus X| < 2|X|$, and let $Y = X \cup N_{G'}(X)$ so that $|Y| \leq 3|X| \leq \frac{1}{20}\epsilon'n$. Then by the minimum degree condition we know that $e_{G'}(Y) \geq \frac{1}{2}|X|\epsilon'np \geq \frac{1}{8}|Y|\epsilon'np$. Now we will estimate the probability that such event can happen for a set Y with $|Y| = a$. We can restrict the range to $\epsilon'np \leq a \leq \frac{1}{20}\epsilon'n$ since G' has minimum degree at least $\epsilon'np$. The probability that there exists a set of size a which spans at least $\frac{1}{8}a\epsilon'np$ edges is,

$$\begin{aligned} \binom{n}{a} \binom{a(a-1)/2}{a\epsilon'np/8} p^{\epsilon'np/8} &\leq \left(\frac{en}{a}\right)^a \left(\frac{4ea}{\epsilon'np}\right)^{\epsilon'np/8} p^{\epsilon'np/8} = \left(\frac{en}{a}\right)^a \left(\frac{4ea}{\epsilon'n}\right)^{\epsilon'np/8} \\ &\ll \left(\left(\frac{e}{4}\right)^{\epsilon'np/8}\right)^a \ll n^{-2}. \end{aligned}$$

Summing over all $\epsilon'np \leq a \leq \frac{1}{20}\epsilon'n$ we get that the probability that there is a set violating the assertion of the lemma is $o(1)$. \square

4 Proof of Theorem 1.3

In this section we prove Theorem 1.3. First we need an additional lemma which gives us more properties of a random graph with deleted edges.

Lemma 4.1. *For every integer $l \geq 3$ and $\epsilon > 0$ there exists $C = C(\epsilon)$ such that if $p = Cn^{-1+1/(l-1)}$ then $G = G(n, p)$ a.a.s. has the following properties. Let H be a subgraph of G with maximum degree at most $(\frac{1}{2} - \epsilon)np$, $G' = G - H$ and $v \in V$, then*

$$(a) \text{ For every } 1 \leq i \leq l-2, 2^{-i}(np)^i \leq |N_{G'}^{(i)}(v)| \leq (1+\epsilon)^i(np)^i.$$

$$(b) |N_{G'}^{(l-1)}(v)| \geq (\frac{1}{2} + \frac{\epsilon}{3})n.$$

$$(c) \text{ For every vertex } w \in V \text{ whose distance from } v \text{ is at least } l-2, |N_G^{(l-2)}(v) \cap N_G(w)| \leq \log n.$$

Proof. Let v be a vertex of G and for simplicity of notation let $Y_j = N_{G'}^{(j)}(v)$ for $j = 1, \dots, l-2$.

(a) By using induction we will show that $2^{-i}(np)^i \leq |Y_i| \leq (1+\epsilon)^i(np)^i$ for all $1 \leq i \leq l-2$. For the initial case $i = 1$ by Lemma 3.2 we have $|Y_1| \leq |N_G^{(1)}(v)| \leq (1+\epsilon)np$. By the same lemma we also know that $|N_G^{(1)}(v)| \geq (1-\epsilon)np$. Therefore $|Y_1| \geq |N_G^{(1)}(v)| - |N_H^{(1)}(v)| \geq np/2$. Now assume that we have established the claim up to some $i \leq l-3$ and let us look at the case $i+1$. First notice $(1+\epsilon)^{l-3}(np)^{l-3} \leq (2C)^{l-1}n^{-1}p^{-2}$ so that we can apply Lemma 3.2. Then the upper bound easily follows as $|Y_{i+1}| \leq |N_G(Y_i)| \leq (1+\epsilon)np|Y_i| \leq (1+\epsilon)^{i+1}(np)^{i+1}$ by that lemma and the inductive hypothesis. To obtain the lower bound, we use that $|N_H(Y_i)| \leq \Delta(H)|Y_i|$ and that $|N_G(Y_i)| \geq (1-\epsilon/2)np|Y_i|$ by Lemma 3.2 (where we substitute $\epsilon/2$ instead of ϵ). Therefore,

$$|N_{G'}(Y_i)| \geq |N_G(Y_i)| - |N_H(Y_i)| \geq (1-\epsilon/2)np|Y_i| - (1/2-\epsilon)np|Y_i| = (1/2+\epsilon/2)np|Y_i|.$$

Recall the recursive formula $Y_{i+1} = N_{G'}(Y_i) - Y_i - Y_{i-1}$. By the inductive hypothesis it is easy to check that $|Y_{i-1}| = o(|Y_i|)$. Thus,

$$|Y_{i+1}| \geq |N_{G'}(Y_i)| - |Y_i| - |Y_{i-1}| \geq (1/2 + \epsilon/2)np|Y_i| - o(np|Y_i|) \geq 2^{-i-1}(np)^{i+1},$$

which completes the proof of the first part.

(b) By part (a) we have $2^{-l+2}(np)^{l-2} \leq |Y_{l-2}| \leq (1 + \epsilon)^{l-2}(np)^{l-2}$ and $|Y_{l-3}| \leq (1 + \epsilon)^{l-3}(np)^{l-3}$. Apply Lemma 3.3 to get $|N_{G'}(Y_{l-2})| \geq (1/2 + \epsilon/2)n$. Then

$$|Y_{l-1}| \geq |N_{G'}(Y_{l-2})| - |Y_{l-2}| - |Y_{l-3}| \geq (1/2 + \epsilon/2)n - (2np)^{l-2},$$

and therefore for large enough n , $|Y_{l-1}| \geq (1/2 + \epsilon/3)n$.

(c) Condition on the event that $|N_G^{(i)}(v)| \leq (1 + \epsilon)^i(np)^i$ for all $i = 0, \dots, l-2$ and let $X = \cup_{i=0}^{l-3} N_G^{(i)}(v)$. Notice that so far we only exposed the edges inside $G[X]$ and the edges connecting X to $N_G^{(l-2)}(v)$. Therefore for any vertex $w \notin X$ which is at distance is at least $l-2$ from v , the edges between w and $N_G^{(l-2)}(v)$ are not yet exposed. Thus we can bound the probability that w has degree at least $\log n$ in $N_G^{(l-2)}(v)$ as follows:

$$\left(\frac{|N_G^{(l-2)}(v)|}{\log n} \right) p^{\log n} < \left(\frac{e|N_G^{(l-2)}(v)|p}{\log n} \right)^{\log n} < \left(\frac{e2^{l-2}(np)^{l-2}p}{\log n} \right)^{\log n} = \left(\frac{e2^{l-2}C^{l-1}}{\log n} \right)^{\log n}.$$

Since the last estimate is $o(n^{-2})$, a.a.s. every pair of vertices as above satisfies the claim. \square

Now we are ready to prove Theorem 1.3. First we restate it in a more accurate and general form.

Theorem 4.2. *For every $\epsilon > 0$ there exists C such that if $p \geq Cn^{-1+1/(l-1)}$ then $G(n, p)$ almost surely has local resilience at least $(1/2 - \epsilon)np$ with respect to being pancyclic.*

Proof. By Proposition 3.1 we may assume that $p = Cn^{-1+1/(l-1)}$ where C is taken to be the maximal of the corresponding constants in Theorem 2.3 and Lemma 4.1. Let $G = G(n, p)$, H be a subgraph of maximum degree $\Delta(H) \leq (\frac{1}{2} - \epsilon)np$, and $G' = G - H$. The proof consists of three parts. In each part we will show the existence of short, medium length, and long cycles, respectively, in G' .

Short Cycles. The existence of cycles of length l to $2l-2$ in G' is a direct corollary of Haxell, Kohayakawa and Łuczak's Theorem 2.3.

Medium Length Cycles. Now we show the existence of cycles of length $2l-1$ up to $\frac{1}{320}\epsilon n$. Fix a vertex $v \in V$ and let $Y = N_{G'}^{(l-1)}(v)$. Then by Lemma 4.1 part (b) a.a.s. $|Y| \geq (\frac{1}{2} + \epsilon/3)n$. By applying the Chernoff inequality and then taking the union bound over sets Y of appropriate sizes, we know that a.a.s. $e_G(Y) \geq (1 - \epsilon/6) \binom{|Y|}{2} p \geq \frac{1}{4}|Y|np$. And by the restriction on the maximum degree of H we know that $e_H(Y) \leq \frac{1}{2}(\frac{1}{2} - \epsilon)|Y|np$. Therefore,

$$e_{G'}(Y) \geq e_G(Y) - e_H(Y) \geq \frac{1}{2}\epsilon|Y|np \geq \frac{1}{4}\epsilon n^2 p.$$

Thus by Lemma 2.5, we can find a subgraph $G_1 \subset G'[Y]$ with minimum degree at least $\frac{1}{4}\epsilon np$. Fix any vertex $v_{l-1} \in V(G_1)$ and let $v_i \in N_{G'}^{(i)}(v)$ for $i = 1, \dots, l-2$ be the vertices of a path $vv_1v_2 \dots v_{l-1}$ in G'

from v to v_{l-1} . Delete every vertex in $N_{G'}^{(l-2)}(v_1) \cap N_{G'}^{(l-1)}(v) \subset N_{G'}^{(l-1)}(v)$ except v_{l-1} from G_1 to obtain G_2 . Then by Lemma 4.1 part (c), $\delta(G_2) \geq \delta(G_1) - \log n$, and so for large enough n , G_2 has minimum degree at least $\frac{1}{8}\epsilon np$. Now by Lemma 3.4, G_2 has the property that every subset X of size $|X| \leq \frac{1}{640}\epsilon n$ satisfies $|N_{G_2}(X) \setminus X| \geq 2|X|$. Therefore by Pòsa's rotation-extension Lemma 2.7 we can find a path P of length at least $\frac{1}{320}\epsilon n$ starting at v_{l-1} inside G_2 . Let this path be $P = v_{l-1}w_1 \dots w_{\epsilon' n}$ where $\epsilon' \geq \frac{1}{320}\epsilon$. Finally observe that for any vertex w_t ($t > 0$) there is a path $v z_1 z_2 \dots z_{l-2} w_t$ in G' such that $z_j \in N_{G'}^{(j)}(v)$ for $j = 1, \dots, l-2$. Moreover since we deleted vertices that can be reached from v_1 , $v_j \neq z_j$ for all $1 \leq j \leq l-2$. Thus we have a cycle $vv_1 \dots v_{l-1}w_1 \dots w_t z_{l-2} \dots z_1 v$ which has length $t + 2l - 2$. Since t can be arbitrarily chosen in the range $1 \leq t \leq \epsilon' n$, we are done with the second part of the proof.

Long Cycles. Let $\alpha = \frac{1}{320}\epsilon$. In this part we will show how to find all cycles of length from αn to n in G' . For a fixed integer n^* satisfying $\alpha n \leq n^* \leq n$ choose uniformly at random n^* vertices V^* out of V and let $G^* = G[V^*]$, $H^* = H[V^*]$. Let \mathcal{P} be the graph property of a graph on n^* vertices having local resilience at least $(1/2 - \epsilon/2)n^*p$ with respect to hamiltonicity. We claim that with probability $1 - o(n^{-1})$, $P(G^* \in \mathcal{P} | G) \geq 1/2$. First note that G^* has distribution $G(n^*, p)$ and apply Sudakov and Vu's Theorem 2.1 to get $P(G^* \notin \mathcal{P}) = o(n^{-1})$. Let A be the event in the probability space $G(n, p)$ such that $P(G^* \notin \mathcal{P} | G) \geq 1/2$. Then we have $(1/2)P(A) \leq P(G^* \notin \mathcal{P}) = o(n^{-1})$. Therefore $P(A) = o(n^{-1})$, or in other words $P(G^* \in \mathcal{P} | G) \geq 1/2$ with probability at least $1 - o(n^{-1})$. Let \mathcal{A}_{n^*} be the collection of graphs G having this property.

On the other hand, observe that the degree of a vertex in H^* follows the hypergeometric distribution and thus we can apply Lemma 2.9. Hence for a vertex $v \in V^*$,

$$P(|\deg_{H^*}(v) - (1/2 - \epsilon)n^*p| \geq \epsilon n^*p/2) \leq e^{-\Omega_\epsilon(n^*p)} \leq e^{-\Omega_\epsilon(np)},$$

thus a.a.s. every vertex in V^* has degree at most $(1/2 - \epsilon/2)n^*p$ in H^* . We can conclude that if $G \in \mathcal{A}_{n^*}$ then there exists a set V^* of size n^* such that $G^* \in \mathcal{P}$ and $\Delta(H^*) \leq (1/2 - \epsilon/2)n^*p$. This gives a hamilton cycle inside $G^* - H^*$ which is a cycle of length n^* inside $G' = G - H$.

Finally note that since $G \in \mathcal{A}_{n^*}$ with probability at least $1 - o(n^{-1})$, cycles of length n^* exist with probability at least $1 - o(n^{-1})$ for any fixed n^* by the previous observation. Therefore by taking the union bound we can see that a.a.s. G' simultaneously contains cycles of length n^* for all $\alpha n \leq n^* \leq n$. This concludes the proof. \square

5 Properties of pseudo-random graphs

Here we collect properties of pseudo-random graphs which we will use later to prove Theorem 1.4. The main fact that we use about (n, d, λ) -graphs is the following formula established by N. Alon (see, e.g., [13]) which connects between eigenvalues and edge distribution.

Lemma 5.1. *If $G = (V, E)$ is an (n, d, λ) -graph, then for any $X, Y \subset V$ we have,*

$$\left| e(X, Y) - \frac{d}{n}|X||Y| \right| \leq \lambda \sqrt{|X||Y|}.$$

As in Section 3 we will prove several lemmas that establish some expansion properties of pseudo-random graphs. These lemmas correspond to Lemma 3.2, Lemma 3.3 and Lemma 3.4 in the random graph case.

Lemma 5.2. *Let ϵ, ϵ' be such that $0 \leq 5\epsilon' < \epsilon, k \geq 3$, and let $G = (V, E)$ be an $(n, \epsilon', d, \lambda)$ -graph with $d^{k-1}/n = \omega(n)\lambda^{k-2}$ where $\omega(n) \rightarrow \infty$. Then G has the following property. If H is a subgraph of G with $\Delta(H) \leq (1/2 - \epsilon)d$ and $G' = G - H$, then every set X with $|X| \leq \epsilon n/4$ satisfies $|N_{G'}(X)| \geq \min(\epsilon n/2, \frac{d^2}{4\lambda^2}|X|)$.*

Proof. Let $Y = N_{G'}(X)$ and assume that $|Y| \leq \epsilon n/2$ as otherwise we are done. Since G' has minimum degree at least $(1 - \epsilon')d - \Delta(H) \geq (1 - \epsilon')d - (1/2 - \epsilon)d \geq (1/2 + 4\epsilon/5)d$, we have

$$\begin{aligned} e_{G'}(X, Y) &\geq (1/2 + 4\epsilon/5)d|X| - 2e_G(X) \geq (1/2 + 4\epsilon/5)d|X| - (d|X|^2/n + \lambda|X|) \\ &\geq (1/2 + 4\epsilon/5)d|X| - (\epsilon d/4 + \lambda)|X| \geq (1/2 + \epsilon/2)d|X|. \end{aligned} \quad (2)$$

On the other hand, since $|Y| \leq \epsilon n/2$, we have:

$$e_G(X, Y) \leq \frac{d|X||Y|}{n} + \lambda\sqrt{|X||Y|} \leq (\epsilon/2)d|X| + \lambda\sqrt{|X||Y|}. \quad (3)$$

Therefore by (2),(3) and $e_{G'}(X, Y) \leq e_G(X, Y)$ we have, $(1/2 + \epsilon/2)d|X| \leq (\epsilon/2)d|X| + \lambda\sqrt{|X||Y|}$ which implies $|Y| \geq \frac{d^2}{4\lambda^2}|X|$. \square

Lemma 5.3. *Let $k \geq 3$ and let $G = (V, E)$ be an $(n, \epsilon', d, \lambda)$ -graph with $d^{k-1}/n = \omega(n)\lambda^{k-2}$ where $\omega(n) \rightarrow \infty$. Then for any function $\delta = \delta(n)$ such that $1 \ll \delta \ll d^2/\lambda^2$, G has the following property. If H is a subgraph of G with $\Delta(H) \leq (1/2 - \epsilon)d$ and $G' = G - H$, then every set X with $|X| \geq \delta(\lambda^2/d^2)n$ satisfies $|N_{G'}(X)| \geq (1/2 + \epsilon/2)n$.*

Proof. We only have to verify this for sets of size exactly $\delta(\lambda^2/d^2)n$, so assume for the contrary that there exists $X \subset V$ of size $|X| = \delta(\lambda^2/d^2)n$ which has $|N_{G'}(X)| < (1/2 + \epsilon/2)n$ and define $Y = V \setminus (X \cup N_{G'}(X))$. Since $|X| = o(n)$, we have $|Y| > (1/2 - 2\epsilon/3)n$. Therefore by Lemma 5.1,

$$e_G(X, Y) \geq \frac{d|X||Y|}{n} - \lambda\sqrt{|X||Y|} \geq (1/2 - 2\epsilon/3)\delta(\lambda^2/d)n - \delta(\lambda^2/d)n > (1/2 - \epsilon)\delta(\lambda^2/d)n.$$

On the other hand by the maximum degree restriction, $e_H(X, Y) \leq (1/2 - \epsilon)d|X| = (1/2 - \epsilon)\delta(\lambda^2/d)n$. But since there are no edges between X and Y we must have $0 = e_{G'}(X, Y) \geq e_G(X, Y) - e_H(X, Y) > 0$ which gives us a contradiction. \square

The next lemma proves expansion property for subgraphs of (n, d, λ) -graphs with large minimum degree.

Lemma 5.4. *Let $G = (V, E)$ be an (n, d, λ) -graph with $\lambda = o(d)$, and let G' be a subgraph of G with $\delta(G') \geq \epsilon d$ for some fixed constant $\epsilon > 0$. Then every $X \subset V(G')$ with $|X| \leq \epsilon n/10$ satisfies $|N(X) \setminus X| \geq 2|X|$.*

Proof. Assume to the contrary that there exists a set $X \subset V(G')$ with $|X| \leq \epsilon n/10$ and $|N(X) \setminus X| < 2|X|$. Let $A = X \cup N(X)$ and note that $|A| < 3|X|$. Then by Lemma 5.1,

$$e_G(A) = e_G(A, A)/2 \leq \frac{d}{2n}|A|^2 + \frac{\lambda}{2}|A| \leq |X|(9\epsilon d/10 + 3\lambda)/2.$$

On the other hand, since G' has minimum degree at least ϵd , we have

$$e_G(A) \geq e_{G'}(A) \geq |X|\epsilon d/2$$

which is a contradiction, since $\lambda = o(d)$. \square

6 Proof of Theorem 1.4

In this section we prove Theorem 1.4. As in the random graph case, we need an additional lemma which gives us more properties of a pseudo-random graph with deleted edges.

Lemma 6.1. *Fix $\epsilon > 0, k \geq 3$ and let $G = (V, E)$ be a (n, d, λ) -graph with $d^{k-1}/n = \omega(n)\lambda^{k-2}$ where $\omega(n) \rightarrow \infty$. Let H be a subgraph of G with $\Delta(H) \leq (1/2 - \epsilon)d$, $G' = G - H$ and $v \in V$. Then there exist $l, 1 \leq l \leq \lfloor (k-1)/2 \rfloor$ and sets $X_i(v), Y_i(v)$ for $i = 0, 1, \dots, l$ such that,*

- (a) $X_0(v) = Y_0(v) = \{v\}$, $X_i(v) \cap Y_i(v) = \emptyset, |X_i(v)| = |Y_i(v)|$ for all $i \neq 0$;
- (b) $|X_{i+1}(v)| \geq \frac{d^2}{16\lambda^2}|X_i(v)|$, $|Y_{i+1}(v)| \geq \frac{d^2}{16\lambda^2}|Y_i(v)|$ for all $i = 0, \dots, l-2$ and $|X_i(v)| = |Y_i(v)| \leq (\epsilon/(8k))n$ for all $i = 0, 1, \dots, l$;
- (c) Let $Z_i(v) = \cup_{j=0}^i (X_j(v) \cup Y_j(v))$. Then $X_{i+1}(v) \subset N_{G'}(X_i(v)) \setminus Z_i(v)$, $Y_{i+1}(v) \subset N_{G'}(Y_i(v)) \setminus Z_i(v)$ for all $0 \leq i \leq l-1$.
- (d) $|X_l(v)| = |Y_l(v)| \geq \delta(\lambda^2/d^2)n$ for some function $\delta = \delta(n) \rightarrow \infty$.

Proof. Let $\delta = \delta(n) = \min(d/\lambda, (\omega(n))^{1/2})$ and note that indeed $\delta \rightarrow \infty$. Given a vertex $v \in V$, we will inductively construct sets $X_i = X_i(v), Y_i = Y_i(v)$ satisfying the condition above. Since G' has minimum degree at least $(1/2 + \epsilon)d$, put $d/4$ vertices of $N_{G'}(v)$ into X_1 and put another $d/4$ vertices into Y_1 . Suppose that for some $i \leq \lfloor (k-1)/2 \rfloor - 1$ we have already constructed $X_0, Y_0, \dots, X_i, Y_i$ satisfying conditions (a), (b), (c), (d). Next we show how to construct X_{i+1}, Y_{i+1} . Let $Z_i = \cup_{j=0}^i (X_j \cup Y_j)$ and note that $|Z_i| \leq 2i|X_i| \leq k|X_i|$. If $|X_i| = |Y_i| \geq \delta(\lambda^2/d^2)n$ then define $l = i$ and stop the process. Otherwise $|X_i| = |Y_i| < \delta(\lambda^2/d^2)n \leq (\lambda/d)n = o(n)$ and by Lemma 5.2 we have that $|N(X_i)|, |N(Y_i)| \geq \min(\epsilon n/2, \frac{d^2}{4\lambda^2}|X_i|)$. Thus

$$|N(X_i) \setminus Z_i| \geq \min\left(\epsilon n/2 - k|X_i|, \frac{d^2}{4\lambda^2}|X_i| - k|X_i|\right) \geq \min\left(\epsilon n/4, \frac{d^2}{8\lambda^2}|X_i|\right)$$

and a similar inequality also holds for $N(Y_i)$. Therefore, by splitting the vertices of $N(X_i) \cap N(Y_i)$ between X_{i+1} and Y_{i+1} we can always choose $X_{i+1} \subset N(X_i) \setminus Z_i$ and $Y_{i+1} \subset N(Y_i) \setminus Z_i$ so that $X_{i+1} \cap Y_{i+1} = \emptyset$ and $|X_{i+1}| = |Y_{i+1}| \geq (1/2) \min(\epsilon n/4, \frac{d^2}{8\lambda^2}|X_i|)$. If $\epsilon n/4 \leq \frac{d^2}{8\lambda^2}|X_i|$ then $|X_{i+1}|, |Y_{i+1}| \geq \epsilon n/8$, so stop the process and define $l = i+1$. Otherwise we can make $|X_{i+1}|, |Y_{i+1}| \geq \frac{d^2}{16\lambda^2}|X_i|$ and continue. Note that (b) holds in this case. If the process does not terminate after constructing $X_1, \dots, X_{\lfloor (k-1)/2 \rfloor}$ and $Y_1, \dots, Y_{\lfloor (k-1)/2 \rfloor}$ then by property (b) we get that $\frac{d}{4}(\frac{d^2}{16\lambda^2})^{\lfloor (k-1)/2 \rfloor - 1} \leq |X_{\lfloor (k-1)/2 \rfloor}| < \delta \frac{\lambda^2}{d^2}n$. This implies:

$$\delta \frac{\lambda^2}{d^2}n > \frac{d}{4} \left(\frac{d^2}{16\lambda^2} \right)^{\lfloor (k-1)/2 \rfloor - 1} \geq \frac{d}{4} \left(\frac{d^2}{16\lambda^2} \right)^{k/2 - 2} = \frac{d^{k-3}}{4^{k-3}\lambda^{k-4}} = \frac{\omega(n)}{4^{k-5}} \cdot \frac{\lambda^2}{d^2}n$$

which is a contradiction, since $\delta \ll \omega(n)$. Finally note that we can always shrink final sets X_l, Y_l so that they become smaller than $(\epsilon/(8k))n$. Since $|X_{l-1}| = |Y_{l-1}| < \delta(\lambda^2/d^2)n \ll (\epsilon/(8k))n$, (b) holds for all $i = 0, 1, \dots, l$. Thus we can find sets as claimed. \square

We are ready to prove Theorem 1.4. First we restate it here with more quantifiers.

Theorem 6.2. Fix $\epsilon > 0$ and let k be either 3 or an even integer satisfying $k \geq 4$, and let $G = (V, E)$ be a (n, d, λ) -graph satisfying $d^{k-1}/n = \omega(n)\lambda^{k-2}$ where $\omega(n) \rightarrow \infty$. Then for large enough n , G has local resilience at least $(1/2 - \epsilon)d$ with respect to containing cycles of length t for $k \leq t \leq n$.

Proof. Let H be a subgraph of G with $\Delta(H) \leq (1/2 - \epsilon)d$ and let $G' = G - H$. If $d > (1 - \epsilon)n$, then G' has minimum degree larger than $(1/2 + \epsilon)d > (1/2 + \epsilon)(1 - \epsilon)n > n/2$ for small enough $\epsilon > 0$. Hence by Bondy's theorem, mentioned in the introduction, G' is pancyclic. Assume therefore that $d \leq (1 - \epsilon)n$. This implies that $\lambda = \Omega(\sqrt{d})$. Indeed, let A be the adjacency matrix of G and $d = \lambda_1, \dots, \lambda_n$ be its eigenvalues. The trace of A^2 is the number of ones in A , which implies that

$$2|E(G)| = nd = \text{Tr}(A^2) = \sum_{i=1}^n \lambda_i^2 \leq d^2 + (n-1)\lambda^2.$$

Solving the above inequality for λ establishes the claim.

A proof of the theorem consists of three parts. In each part we will show the existence of short, medium length and long cycles respectively.

Short Cycles. For $k = 3$, the existence of 3-cycles is a direct corollary of Sudakov, Szabó and Vu's Theorem 2.4. Also in this case we have that $d^3/\lambda^2 \geq d^2/\lambda \gg n$. Therefore for $k = 3$ the existence of cycles of length $4, \dots, n$ follows from the proof of case $k = 4$. So from now on we assume that $k = 2k'$ is even. Since $\lambda = \Omega(\sqrt{d})$, we have $d^{k-1} = \omega(n)\lambda^{k-2}n = \omega(n)\Omega(d^{k/2-1})n$. Therefore $d \gg n^{2/k}$ and by Bondy and Simonovits' Theorem 2.6 G' must have a k -cycle.

Medium Length Cycles. The next step is to prove the existence of cycles of length from $k+1$ up to $\epsilon n/20$. Fix a vertex v and apply Lemma 6.1 to find sets $X_1 = X_1(v), \dots, X_l = X_l(v), Y_1 = Y_1(v), \dots, Y_l = Y_l(v)$ where $l \leq k' - 1$ with $|X_l| = |Y_l| \geq \delta(\lambda^2/d^2)n$ and $|X_i| = |Y_i| \leq (\epsilon/(8k))n$ for all $i = 1, \dots, l$. Then $|\cup_{i=0}^l X_i \cup Y_i| \leq \epsilon n/4$. By Lemma 5.3 we know that $|N_{G'}(X_l)| \geq (1/2 + \epsilon/2)n$ and so if we let $Z = N_{G'}(X_l) \setminus (\cup_{i=0}^l X_i \cup Y_i)$ then $|Z| \geq (1/2 + \epsilon/4)n$. Since $\lambda = o(d)$, by Lemma 5.1, we have $e_G(Z) \geq d|Z|^2/(2n) - \lambda|Z|/2 \geq d|Z|/4$. On the other hand $e_H(Z) \leq (1/2 - \epsilon)d|Z|/2$. Hence $e_{G'}(Z) \geq e_G(Z) - e_H(Z) \geq \epsilon d|Z|/2$. This implies by Lemma 2.5 that inside $G'[Z]$ we can find a subgraph $G_1 \subset G'$ which has minimum degree at least $\epsilon d/2$. Then using Lemma 5.1 it is easy to check that $|V(G_1)| \geq \epsilon n/3$ and so we can choose a set $W \subset V(G_1) \subset Z$ of size $\epsilon n/8$. Then by Lemma 5.3, we have that both $|N_{G'}(W)|, |N_{G'}(Y_l)| \geq (1/2 + \epsilon)n$. Therefore the set $(N_{G'}(W) \cap N_{G'}(Y_l)) \setminus (\cup_{i=0}^l X_i \cup Y_i)$ has size at least $2\epsilon n - \epsilon n/4 > 0$. In particular, there must exist a vertex $p \in (N_{G'}(W) \cap N_{G'}(Y_l)) \setminus (\cup_{i=0}^l X_i \cup Y_i)$, and let $y_l \in Y_l, w \in W$ be neighbors of p in G' . Since $y_l \in Y_l$, by the definition of Y_l , there exists a path $vy_1y_2 \dots y_l$ in G' from v to y_l such that $y_i \in Y_i$ for $i = 1, \dots, l$. If $p \in V(G_1)$ then let $G_2 = G_1 \setminus \{p\}$ and note that G_2 has minimum degree at least $\epsilon d/4$. Now by Lemma 5.4 every set $X \subset V(G_2)$ of size $|X| \leq \epsilon d/40$ satisfies $|N_{G_2}(X) \setminus X| \geq 2|X|$. Then by Pòsa's rotation-extension Lemma 2.7 we know that there exists a path $P = v_0v_1 \dots v_t$ starting at $v_0 = w$ which has length at least $t \geq \epsilon n/20$ inside G_2 . For an arbitrary $v_i \in P, i \geq 0$ since $v_i \in V(G_2) \subset N_{G'}(X_l)$, there is a path $vx_1 \dots x_lv_i$ in G' such that $x_i \in X_i$ for $i = 1, \dots, l$. Thus we have a cycle $vx_1x_2 \dots x_lv_iv_{i-1} \dots v_0py_ly_{l-1} \dots y_1v$ which has length $2(l+1) + i + 1 \leq k + i + 1$. Since i can be arbitrarily chosen in the range $0 \leq i \leq \epsilon n/20$, we are done with the second part of the proof.

Long Cycles. The final step is to prove the existence of cycles of length $\epsilon n/20$ to n . Pick $\epsilon/20 \leq \alpha \leq 1$ such that $n^* = \alpha n$ is an integer. Let $V^* \subset V$ be a set of size n^* chosen uniformly at random and

$G^* = G[V^*], H^* = H[V^*]$ be the induced subgraph of G, H respectively. Since $d \gg n^{2/k}$, by the concentration of the hypergeometric distribution (Lemma 2.9), for every vertex $v \in V^*$ we have that

$$P(\deg_{H^*}(v) \geq (1/2 - \epsilon/2)\alpha d) \leq e^{-\Omega_\epsilon(\alpha d)} \leq e^{-\Omega_\epsilon(n^{2/k})}.$$

Similarly, with probability $1 - o(n^{-1})$, the graph G^* has minimum degree $(1 - \epsilon/6)\alpha d$ and therefore if n is large enough, for every $\epsilon n/20 \leq n^* \leq n$ there exists a choice of V^* where $\Delta(H^*) \leq (1/2 - \epsilon/2)\alpha d$ and $\delta(G^*) \geq (1 - \epsilon/6)\alpha d$. Moreover since G^* is an induced subgraph of G , its edge distribution is still governed by the estimate from Lemma 5.1. Therefore G^* is an $(\alpha n, \epsilon/6, \alpha d, \lambda)$ -graph. Now by Sudakov and Vu's Theorem 2.2, for large enough n , G^* has local resilience at least $(1/2 - \epsilon/2)\alpha d$ with respect to being hamiltonian. Thus for the choice of V^* as above, $G^* - H^*$ must contain a hamilton cycle which is a cycle of length n^* in G' . This concludes the proof. \square

7 Concluding Remarks

7.1 Cycles through a given vertex

In this paper we found cycles inside a subgraph of random and pseudo-random graphs. We proved that under certain conditions there exist cycles of various lengths *somewhere* in the subgraph. In fact, we can find most of these cycles even if we fix a vertex v and insist that a cycle of desired length passes through this vertex.

Theorem 1.3 says that for any fixed integer $l \geq 3$, if $p \gg n^{-1+1/(l-1)}$ then $G(n, p)$ almost surely has local resilience $(1/2 + o(1))np$ with respect to containing cycles of length t for $l \leq t \leq n$. By carefully examining the proof, one can realize that when finding middle length and long cycles, we can insist on the cycle to pass through a fixed vertex. Thus we have that for any fixed vertex v , there is a cycle of length t for $2l - 1 \leq t \leq n$ which passes through v . Observe in addition that for every odd integer $l \leq t < 2l - 1$, we cannot guarantee a cycle of length t through a fixed vertex v as can be seen using Lemma 3.2. Let $G = G(n, p)$. This lemma implies that for any fixed vertex v , $|N_G^{(i)}(v)| = o(n)$ for all $1 \leq i \leq l - 2$. Therefore typically the degrees inside $N_G^{(i)}(v)$ are all $o(np)$, and we can delete every edge inside these sets without violating the maximum degree condition on H to get a graph $G - H$ which does not contain a cycle of length t through v .

Similarly, Theorem 1.4 says that if k is either 3 or an even integer satisfying $k \geq 4$ and $G = (V, E)$ is a (n, d, λ) -graph satisfying $d^{k-1}/n \gg \lambda^{k-2}$, then G has local resilience $(1/2 + o(1))d$ with respect to containing cycles of length t for $k \leq t \leq n$. Again even if we fix a vertex we can force all the middle length and long cycles to contain it. Namely, for any fixed vertex v , there exists a cycle of length $k + 1$ up to n passing through v .

7.2 Paths through a given pair of vertices

Another possible and rather straightforward extension of our results is to show that random and pseudo-random graphs are locally resilient with respect to the following property: for every given pair of vertices u, v and for every given length $l \geq t$ (where t is a constant depending on our choice of parameters, quite similarly to the situation in Theorems 1.3 and 1.4) there is a path of length l between u and v . We do not provide much details here, but here is a very short sketch of the argument. For medium length paths (between t and δn , for some constant $\delta > 0$) the proof is obtained by a rather

trivial modification of the corresponding proofs for medium length cycles in Theorems 1.3 and 1.4. For example, in the pseudo-random case, instead of growing sets $X_i(v)$, $Y_i(v)$ as in the proof of Theorem 1.4, we grow disjoint sets $X_i(u)$ and $X_i(v)$ till they reach substantial size, and then find a subgraph G_1 with large minimum degree on at least $(1/2 + \delta)n$ vertices in the neighborhood of $X_i(u)$. Since $|V(G_1)| \geq (1/2 + \delta)n$, the set $X_i(v)$ has a neighbor w in G_1 . Due to expansion properties of G_1 , there is a path of linear length in it starting from w . This path can be used to find paths of medium length between u and v . For paths of linear length, the key is the ability to find a Hamilton path through a given pair of vertices u, v in an edge deleted random or pseudo-random graph. Here we can argue as follows. First, if $e = (u, v) \notin E(G) \setminus E(H)$ (where G is the original (pseudo-)random graph, and H is the graph of deleted edges meeting the imposed condition on maximum degree), add e to the graph; clearly nothing really changes in its edge distribution. Then, find a path P of linear length with e in somewhere in the middle (i.e., some sizable distance from both ends), and then grow P and close it to a Hamilton cycle through rotations and extensions as usually, each time forbidding to touch an interval of constant length surrounding P ; our expansion assumptions enable easily to meet this restriction. The so obtained Hamilton cycle C is guaranteed to contain e . Finally, omit e from C , thus getting a Hamilton path between u and v .

7.3 Open Problems

We believe that Theorem 2.4 can be extended (with appropriate adjustments) to cycles of an arbitrary but fixed odd length. More specifically, it is plausible that for an odd $k \geq 5$, if G is an (n, d, λ) -graph and $d^{k-1}n / \gg \lambda^{k-2}$, then the local resilience of G with respect to containing a cycle of length k is $(1/2 - o(1))d$. The validity of this conjecture would allow to extend the assertion of Theorem 1.4 to all $k \geq 3$.

A more natural generalization of Theorem 2.4 (actually, of its original global resilience form as in [15]) is the following conjecture.

Conjecture 7.1. *Let $k \geq 5$ be an odd integer and G be a (n, d, λ) -graph satisfying $d^{k-1}/n \gg \lambda^{k-2}$. Then G has global resilience $(1/4 + o(1))nd$ with respect to being C_k -free.*

A Proof of Theorem 2.2

In this appendix, we illustrate how the proof Theorem 2.2 follows from the results of Sudakov and Vu [16]. First we repeat the statement of the theorem here.

Theorem A.1. *Fix ϵ, ϵ' such that $0 \leq 5\epsilon' < \epsilon, k \geq 3$ and let G be an $(n, \epsilon', d, \lambda)$ -graph satisfying $d^{k-1}/n \gg \lambda^{k-2}$. Then for large enough n , G has local resilience at least $(1/2 - \epsilon)d$ with respect to being hamiltonian.*

As mentioned above, this theorem is not part of the original paper. In fact, they proved the following theorem.

Theorem A.2. *Let $\epsilon > 0$ be fixed and G be a (n, d, ϵ) -graph such that $d/\lambda > \log^2 n$. Then for large enough n , G has local resilience at least $(1/2 - \epsilon)d$ with respect to being hamiltonian.*

Unfortunately we cannot directly apply this result as our graph is not regular and we don't have a bound on d/λ . But Sudakov and Vu proved this result as a corollary of the following two results which can be modified to work in our situation,

Theorem A.3. *For any fixed $\epsilon > 0$ and sufficiently large n , the following holds. Let G be a connected graph of order n such that every subset U of G of size at most $n/\log^4 n$ satisfies $|N_G(U)| \geq \frac{\log^4 n}{15} \cdot |U|$ and every subset W of size at least $n/\log^3 n$ has $|N_G(W)| \geq \frac{1+\epsilon}{2}n$. Then G contains a Hamilton cycle.*

Lemma A.4. *For any fixed $\epsilon > 0$ and sufficiently large n the following holds. Let G be a (n, d, λ) -graph with $d/\lambda > \log^2 n$ and let H be a subgraph of G with maximum degree at most $(1/2 - \epsilon)d$. Then the graph $G' = G - H$ is*

- *connected;*
- *every subset U of G' of size at most $n/\log^4 n$ satisfies $|N_{G'}(U)| \geq \frac{\log^4 n}{15} \cdot |U|$.*
- *every subset W of size at least $n/\log^3 n$ has $|N_{G'}(W)| \geq \frac{1+\epsilon}{2}n$.*

By applying Lemma 5.2 and Lemma 5.3 we have the following result similar to Lemma A.4.

Lemma A.5. *Fix ϵ, ϵ' such that $0 \leq 5\epsilon' < \epsilon, k \geq 3$ and let G be an $(n, \epsilon', d, \lambda)$ -graph satisfying $d^{k-1}/\lambda^{k-2} \gg n$ and let H be a subgraph of G with maximum degree at most $(1/2 - \epsilon)d$. Then the graph $G' = G - H$ is*

- *connected;*
- *every subset U of G' of size at most $(\lambda^2/d^2)n$ satisfies $|N_{G'}(U)| \geq \frac{\epsilon d^2}{4\lambda^2} \cdot |U|$.*
- *for an arbitrary function $\delta(n)$ growing to infinity, every subset W of size at least $\delta(n)(\lambda^2/d^2)n$ has $|N_{G'}(W)| \geq \frac{1+\epsilon}{2}n$.*

Therefore we only need to prove the following theorem which is a variant of Theorem A.3.

Theorem A.6. *Let $\epsilon > 0, k \geq 3$ be fixed $d^{k-1}/\lambda^{k-2} \gg n$. Then for sufficiently large n the following holds. Let G be a connected graph of order n such that every subset U of G of size at most $(\lambda^2/d^2)n$ satisfies $|N_G(U)| \geq \frac{\epsilon d^2}{4\lambda^2} \cdot |U|$ and every subset W of size at least $\delta(n)(\lambda^2/d^2)n$ has $|N_G(W)| \geq \frac{1+\epsilon}{2}n$. Then G contains a Hamilton cycle.*

We omit the proof which is a word by word translation of the proof of Theorem A.3. (Actually we are in a more simple situation since we only need $k/2$ rotations compared to $\log n / \log \log n$ as in the original proof.)

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